Week One

1 Propositions

We begin the study of how mathematics is done by learning some elementary mathematical logic. Our initial focus is on a particular sort of sentence, called a *proposition*. A proposition is simply a sentence that has a *truth value*, which is to say that a proposition is unambiguously either true or false. Generally speaking, mathematical systems are based on *axioms*, which are accepted as true and are therefore propositions. The body of a mathematical system is then filled out with *theorems*, which follow from the axioms by means of *deduction*, or *inference*. Example 1.1 in Gerstein (on p.3) presents a nice little axiomatic system. Another useful type of proposition is the *conjecture*, which is a proposition whose truth value is unknown. For an example, see Example 1.2b in Gerstein. Conjecture plays a vital role in the development of any mathematical system. But our immediate goal here is to gain a working understanding of the process of deduction. To do this, we need tools for constructing propositions and for analyzing their truth values.

2 Connectives and Truth Tables

We begin with the fundamental logical *connectives* called negation, disjunction, and conjunction. Let P and Q denote some anonymous propositions. Either or both of these might be *compound* propositions, in the sense that they themselves are the result of applying connectives to subpropositions. A proposition not so constructed is called *atomic*.

2.1 Negation

The *negation* of P is the proposition whose truth value is exactly opposite that of P. We refer to this as "not P", denoted by $\neg P$. To make the definition crystal clear, we employ a *truth table*. The table for negation looks like this:

$$\begin{array}{c|c} P & \neg P \\ \hline F & T \\ T & F \end{array}.$$

Note that, since P is a proposition, P must be either true or false, and it follows that $\neg P$ must be either false or true. For example, let P be "$3 < 2$". Then $\neg P$ is "$3 \geq 2$". Clearly, exactly one of these is true.

2.2 Disjunction

Our next connective is *disjunction*, which corresponds to the English "or" in its inclusive sense. The notation most commonly used to represent "P or Q" is $P \vee Q$. The official definition is given by the following truth table:

$$\begin{array}{cc|c} P & Q & P \vee Q \\ \hline F & F & F \\ F & T & T \\ T & F & T \\ T & T & T \end{array}.$$

Examples:

1. In algebra, a weak inequality such as "$2 \leq 3$" is an abbreviation representing a disjunction, in this case "$2 < 3$ or $2 = 3$". This particular disjunction is true.

2. The disjunction, "NPS is in Monterey or MA1025 is a mathematics course" is true. Not very interesting, perhaps, but nevertheless true.

Be careful not to confuse the disjunction of two propositions with the so-called *exclusive or*, which is true when precisely one of its arguments is true. In mathematical speaking and writing, the exclusive or must be explicitly stated if it is intended.

2.3 Conjunction

Our third connective is *conjunction*, which corresponds to the English "and". The most common notation for "P and Q" is $P \wedge Q$. The definition is once again delivered by truth table:

$$\begin{array}{cc|c} P & Q & P \wedge Q \\ \hline F & F & F \\ F & T & F \\ T & F & F \\ T & T & T \end{array}.$$

Example: A "sandwich" inequality of the form $x < y < z$ is an abbreviation for the conjunction, "$x < y$ and $y < z$". An instance of this form is $2 < 1 < 3$, which is false. The instance $2 < e < 3$ is true, provided that e represents the base of the natural exponential function.

3 Implication

We now come to implication. Consider the proposition, $\neg(P \wedge \neg Q)$, which is expanded in the truth table below:

P	Q	$\neg(P \wedge \neg Q)$
F	F	T
F	T	T
T	F	F
T	T	T

.

It turns out that this is a remarkably useful form, so useful in fact that it has been abbreviated as $P \Rightarrow Q$, often pronounced "P implies Q". This expression is called an *implication* or a *conditional proposition*. It is often a source of confusion and annoyance because the word, "implies", carries with it a connotation of causality that is misplaced in the sense that $P \Rightarrow Q$ can be true with P false or with P true.

In the implication $P \Rightarrow Q$, P is called the *hypothesis* (or *antecedent*) and Q is the *conclusion* (or *consequent*). Try to become comfortable with the twin ideas that (a) if the hypothesis is false, the implication is true, and (b) if the conclusion is true, the implication is true. These, especially (a), often take students by surprise. An implication with a false hypothesis is called *vacuously true*, while an implication with a true conclusion is called *trivially true*.

Here are some examples:

1. The implication, "If $2 < 1$ then $2 + 2 = 2$," is vacuously true, in spite of the fact that its conclusion is false.

2. The implication, "If $1 < 2$ then $2 + 2 = 4$," is trivially true. The fact that its hypothesis is true is irrelevant.

3. The implication, "If $1 < 2$ then pigs have wings," is false, as far as we know.

4. The implication, "If $x \geq 0$ and $x < y$ then $x^2 < y^2$" is provably true for any real numbers x and y.

In practice, there are many pronunciations of $P \rightarrow Q$. Here is a nonexhaustive list:

1. P implies Q

2. If P, then Q

3. Q if P

4. P only if Q

5. P is sufficient for Q

6. Q is necessary for P

The multiplicity of pronunciations is not intended to confuse. Mathematical writing is, after all, writing. It's nice to have several ways to say the same thing. On the other hand, the existence of so many alternatives can indeed be confusing, especially to the beginner. This is the tip of the iceberg of translation between the symbolic language of logic and a natural language such as English. The symbolic language is precise, while the natural language is loaded with ambiguity and nuance. This makes the process of translation challenging.

3.1 Variations on the Theme: The Converse and Contrapositive

Given the implication $P \to Q$, we commonly make use of two variations. The *converse* of $P \to Q$ is $Q \to P$, while the *contrapositive* of $P \to Q$ is $\neg Q \to \neg P$. Here they are together in a truth table:

P	Q	$P \to Q$	$Q \to P$	$\neg Q \to \neg P$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	T	T

Note that the columns for $P \to Q$ and $\neg Q \to \neg P$ are identical. This is a phenomenon called logical equivalence, and will be taken up later.

4 Exercises

Exercises for week one, from Gerstein: In §1.1, exercises 1, 4. In §1.3, exercises 1, 2, 5, 11, 17.